DISCRETE MATHEMATICS

Belyi functions for Archimedean solids

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Without doubt the authentic type of these figures exists in the mind of God the Creator and shares His eternity.

J. Kepler [17] (cited in [8])

Abstract

The notion of a Belyi function is a main technical tool which relates the combinatorics of maps (i.e., graphs embedded into surfaces) to Galois theory, algebraic number theory, and the theory of Riemann surfaces. In this paper we compute Belyi functions for a class of semi-regular maps which correspond to the so-called Archimedean solids. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

La notion de fonction de Belyi est un outil technique qui relie la combinatoire des cartes (c'est-à-dire, des graphes plongés sur des surfaces) avec la théorie de Galois, la théorie des nombres algébriques et la théorie des surfaces de Riemann. Dans cet article nous calculons les fonctions de Belyi pour une classe des cartes semi-regulières, correspondant à ce qu'on appelle les solides d'Archimède. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

The works of Archimedes, and that of G.V. Belyi are separated by 23 centuries. The latter represent one of the most recent advances in Galois theory, known (even in the English literature) under the French name of the theory of 'Dessins d'Enfants' (which means 'Children's drawings'), see for example [27]. The theory of 'Dessins d'Enfants' was called so by its founder Alexandre Grothendieck [14]. The core notion of the theory is that of a *Belyi function* [3]. In the planar case (which is the only case we consider here) a Belyi function is a rational function defined on the Riemann complex sphere which has at most three critical values, namely, 0, 1 and ∞. A pre-image of the segment $[1,\infty]$ under such a function is a planar (hyper)map. The coefficients of

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such a function are algebraic numbers. This leads to an action of the absolute Galois group on planar maps. See details in Section 2, and for a more extended presentation see [27].

An explicit computation of a Belyi function corresponding to a given map is reduced to a solution of a system of algebraic equations. It may turn out to be extremely difficult. To give an idea of the level of difficulty, we mention that our attempts to compute Belyi functions for some maps with only six edges took us several months, and the result was achieved only after using some advanced Gröbner bases software [11] and numerous consultations given by its author J.C. Faugère (see [23]). The situation changes dramatically when the map in question possesses rich symmetries. Thus, for Platonic solids the Belyi functions (under other name) were computed by Felix Klein as early as in 1875 (see, for example, [19, Sections 10–14 of Chapter II; 7]). Our aim is to continue this study and to compute Belyi functions for a class of semi-regular maps. There are many possible candidates for such a class; for this paper we have chosen the class that is often called Archimedean solids; see details in Section 3. Our biggest 'accomplishment' (if we do not take into account the infinite series of prisms and antiprisms) is a map of 180 edges.

Of course, we were attracted by the subject not only because we wished to 'beat the records', not even because of its geometric beauty. We hope to extract from our computations some interesting facts concerning explicit construction of Galois groups and other related topics; see in this respect [20,21], which examine only one of the Archimedean figures, namely, the truncated icosahedron. The connection of regular and semi-regular maps to Galois theory is after all not so surprising, if one recalls the important role played in this theory by the icosahedron [19].

2. Belyi functions

For more details concerning Belyi functions see [29,7,28].

Let $\bar{\mathbb{C}}$ denote the Riemann complex sphere, and let $f:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ be a rational function. For almost all $w\in\bar{\mathbb{C}}$ the equation $f(z)=w$ has the same number n of solutions; n is called the degree of f ($\deg f=n$). Those values of $w\in\bar{\mathbb{C}}$ for which the equation has less than n solutions (that is, some of the solutions are multiple) are called *critical values*. The multiple roots $z\in\bar{\mathbb{C}}$ of the equation $f(z)=w$ are called *critical points*.

Definition 2.1 (*Belyi function*). A rational function $f:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ is called a *Belyi function* if all its critical values are in $\{0,1,\infty\}$.

Definition 2.2 (*Clean Belyi function*). A Belyi function f is called *clean* if all the points z such that $f(z)=1$ are critical points of multiplicity 2.

If f is a clean Belyi function, the set $f^{-1}([1,\infty])$ is a *planar map*; for an arbitrary Belyi function it is a *planar hypermap* (for the notion of a hypermap see [6]). This means that it is a connected graph drawn on the Riemann sphere of the variable z, and the complement of this graph is a disjoint union of sets homeomorphic to the open disk; these sets are called *faces* of the map. The following additional information is useful for understanding the relation between the Belyi function and the map:

- The vertices of the map are the points of $f^{-1}(\infty)$, the vertex degree being equal to the multiplicity of the corresponding root of the equation $f(z) = \infty$.
- Inside each face there is a unique root of the equation $f(z)=0$, its multiplicity being equal to the face degree. We call this point the *center* of the face.
- The 'open' edges of the map are disjoint intervals of the set $f^{-1}([1,\infty[)$. Inside each edge there is a root of the equation $f(z) = 1$. We call this point the *middle point* of the edge. In the case of a map this root always has multiplicity 2; in the case of a hypermap its multiplicity may be arbitrary.
- Besides the points mentioned above, there are no other critical points of f.
- If we join the center of each face to the vertices and middle points adjacent to this face, we obtain a *canonical triangulation* of the (hyper)map. We call a triangle *positive* (resp. *negative*), if its vertices taken in the counter clockwise direction are labeled as $0, 1, \infty$ (resp. $0, \infty, 1$). The image under the Belyi function of all positive triangles is the upper half-plane, and for the negative ones, the lower half-plane.

Theorem 2.3 (Riemann's existence theorem). *For any hypermap the corresponding Belyi function exists and is unique up to a linear fractional transformation of the variable z.*

Remark 2.4. The above theorem is true not only for 'polyhedral' maps but also for arbitrary maps: loops and multiple edges are allowed, as well as vertices of degree 1 and 2. The only condition is that the corresponding graph must be connected.

The same map may be drawn on the plane in infinitely many ways. But one of its geometric forms is distinguished: it is the form given as the pre-image of the segment $[1,\infty]$ via the Belyi function corresponding to the map. We will call this particular drawing of the map its *dessin d'enfant*.

Example 2.5. The following function, which goes back to Felix Klein [19], is the Belyi function for the dodecahedron (see Fig. 1):

$$f_{\text{dodeca}}(z) = 1728\,\frac{(z^{10} - 11\,z^5 - 1)^5 z^5}{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}.$$

The roots of the polynomial $C(z) = z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1$ in the denominator are the coordinates of the 20 vertices of the dodecahedron; degree 3 in the denominator $C(z)^3$ reflects the fact that all the vertices are of degree 3. The roots of the polynomial $A(z) = (z^{10} - 11\,z^5 - 1)z$ in the numerator are the centers of the faces, all of them of degree 5. We 'see' here only 11 faces; the 12th one is situated at ∞. Its degree is also 5, because the total degree of the numerator is 55, and that of the denominator is 60.

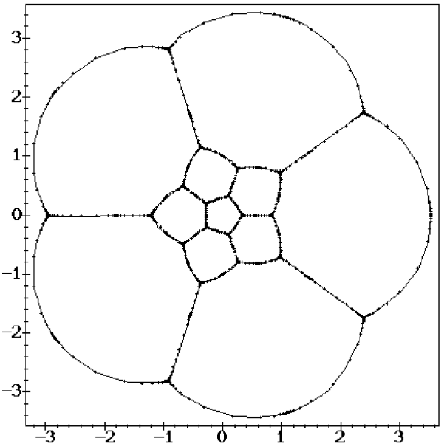

Fig. 1. Dessin d'enfant of a dodecahedron.

Finally, if we compute $f_{\mathrm{dodeca}}(z) - 1$, we find

$$f_{\mathrm{dodeca}}(z) - 1 = -\frac{(z^{10}+1)^2(z^{20} - 522\,z^{15} - 10006\,z^{10} + 522\,z^5 + 1)^2}{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}.$$

Thus, all the 30 roots of the equation $f_{\mathrm{dodeca}}(z) = 1$ are of multiplicity 2, and they are situated at the roots of the polynomial $B(z) = (z^{10}+1)(z^{20} - 522\,z^{15} - 10006\,z^{10} + 522\,z^5 + 1)$. These are the edge middle points.

For completeness one must also verify that there are no other critical points of f_{dodeca}, that is, that $f'_{\mathrm{dodeca}}(z) = 0$ implies $f_{\mathrm{dodeca}}(z) \in \{0, 1, \infty\}$.

Belyi functions for the other Platonic solids are given below.

Tetrahedron (Dessin d'enfant: Fig. 2)

$$f_{\mathrm{tetra}}(z) = -64\frac{(z^3+1)^3}{(z^3-8)^3 z^3}, \qquad f_{\mathrm{tetra}}(z) - 1 = -\frac{(z^6 + 20\,z^3 - 8)^2}{(z^3-8)^3 z^3}.$$

Cube (Dessin d'enfant: Fig. 3)

$$f_{\mathrm{cube}}(z) = -108\frac{(z^4+1)^4 z^4}{(z^8 - 14\,z^4 + 1)^3}, \qquad f_{\mathrm{cube}}(z) - 1 = -\frac{(z^{12} + 33\,z^8 - 33\,z^4 - 1)^2}{(z^8 - 14\,z^4 + 1)^3}.$$

Remark 2.6. It is obvious that if f is a Belyi function for a map, then $1/f$ is a Belyi function for the dual map. Therefore, for the octahedron and for the icosahedron

$$f_{\mathrm{octa}}(z) = \frac{1}{f_{\mathrm{cube}}(z)}, \qquad f_{\mathrm{icosa}}(z) = \frac{1}{f_{\mathrm{dodeca}}(z)}.$$

However, if the center of the 'outer' face of the initial map was situated at ∞, which is only natural, then for the dual map it is one of its vertices which is situated at ∞, and that is not so convenient if we want to draw the dessin d'enfant. Therefore, in practice, this simple transformation needs a certain 'remake', which is a linear fractional transformation of the variable z. This operation may not be entirely harmless, as the

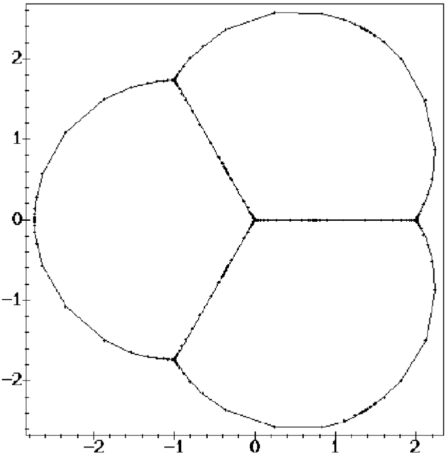

Fig. 2. Tetrahedron.

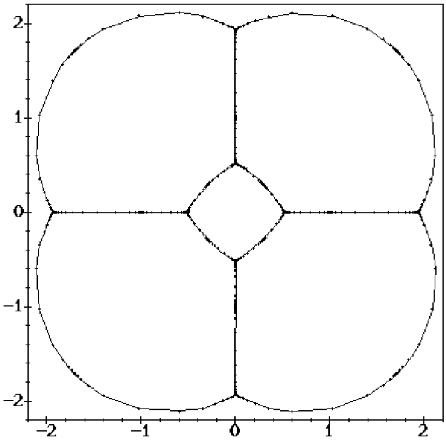

Fig. 3. Cube.

coefficients of the transformation may involve some irrational algebraic numbers even if the coefficients of the original function are rational.

The computation of the above Belyi functions is a story of a rediscovery. First we computed them using a kind of a 'brute force' approach, with very few computational shortcuts. Then we have found them in [7], and, after some additional hints of our colleagues, in [19]. Felix Klein called these functions 'fundamental rational functions', and computed them without any advanced software, using the theory of invariants. Finally, we have found out that the easiest way to compute them consists of using some very classical interrelations between Platonic solids:

1. The Belyi function of the tetrahedron may be easily computed by hand, see the discussion of the truncated tetrahedron below.
2. The middle points of the six edges of a regular tetrahedron are the six vertices of a regular octahedron (Fig. 4). This gives us immediately the Belyi function for an octahedron, and hence for a cube.
3. There is a well-known construction of a cube inscribed in a dodecahedron (see, for example, [4, Section 12.5.5.1]), which gives that 8 out of 20 vertices of a regular dodecahedron are the vertices of a cube. This leads to an easy way of obtaining the Belyi function for a dodecahedron, and hence for an icosahedron (Fig. 5).

It is this interplay of geometric and algorithmic ideas that makes the whole subject so fascinating.

Remark 2.7. As we were informed by Borwein [5], the Belyi function of the tetrahedron is related to his cubic parametrization of Kline's absolute invariant for the full modular group. This relation also involves some theta function identities. For more details see his paper in this volume.

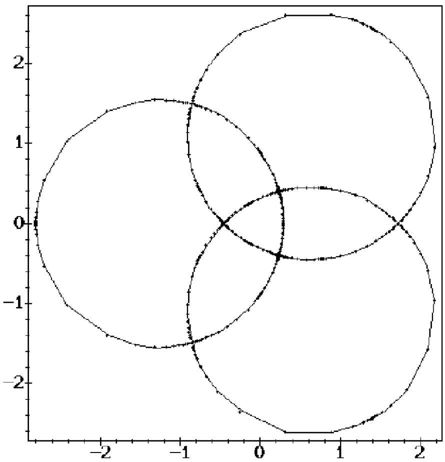

Fig. 4. Octahedron.

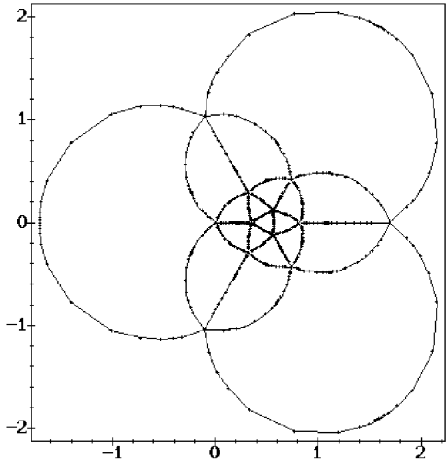

Fig. 5. Icosahedron.

3. Archimedean polyhedra

In order to introduce Archimedean polyhedra, consider planar maps without loops and multiple edges, and also without vertices of degree 1 and 2. For a vertex of degree k let us call its *type* the list of numbers $(f_1, f_2, \ldots, f_k)$, where f_i are the degrees of the adjacent faces taken in the counter-clockwise direction around the vertex. Two such lists are *equivalent* if one can be obtained from the other by making (a) a cyclic shift, and (b) the inversion of the order (which means the change of orientation at this vertex).

Definition 3.1 (*Archimedean map*). A map is called *Archimedean* if the types of all its vertices are equivalent.

It turns out that, combining the above definition with the Euler formula, we obtain as solutions (a) five Platonic maps; (b) two infinite series of maps (prisms and antiprisms), and (c) 14 other solutions. All solutions except Platonic maps are listed in Table 1. Here V, F and E are the number of vertices, faces and edges respectively, and notation '$20 : 3 + 30 : 4 + 12 : 5$' in the 'face distribution' column means '20 triangles, 30 squares and 12 pentagons'.

All the listed maps have geometric realizations in a form of a polyhedron all of whose faces are regular polygons, and all polyhedral angles are congruent. Such polyhedra are called *Archimedean*.

To give an exhaustive or even reasonably representative bibliography on Archimedean polyhedra would be an impossible task. Even their history is rather complicated. Traditionally, their discovery is attributed to Archimedes (Heron and Pappus refer to his manuscript, which perished in a fire of the library at Alexandria). They were re-discovered (and baptized by their now usual names) by Johannes Kepler [18]; see also

Table 1

	Name	V	Vertex type	F	Face distribution	E
1	n-prism	$2n$	$(4,4,n)$	$n+2$	$n:4+2:n$	$3n$
2	n-antiprism	$2n$	$(3,3,3,n)$	$2n+2$	$2n:3+2:n$	$4n$
3	Truncated tetrahedron	12	$(3,6,6)$	8	$4:3+4:6$	18
4	Truncated cube	24	$(3,8,8)$	14	$8:3+6:8$	36
5	Truncated octahedron	24	$(4,6,6)$	14	$6:4+8:6$	36
6	Truncated icosahedron	60	$(5,6,6)$	32	$12:5+20:6$	90
7	Truncated dodecahedron	60	$(3,10,10)$	32	$20:3+12:10$	90
8	Truncated cuboctahedron	48	$(4,6,8)$	26	$12:4+8:6+6:8$	72
9	Truncated icosidodecahedron	120	$(4,6,10)$	62	$30:4+20:6+12:10$	180
10	Cuboctahedron	12	$(3,4,3,4)$	14	$8:3+6:4$	24
11	Icosidodecahedron	30	$(3,5,3,5)$	32	$20:3+12:5$	60
12	Rhombicuboctahedron	24	$(3,4,4,4)$	26	$8:3+18:4$	48
13	Rhombicosidodecahedron	60	$(3,4,5,4)$	62	$20:3+30:4+12:5$	120
14	Snub cube	24	$(3,3,3,3,4)$	38	$32:3+6:4$	60
15	Snub dodecahedron	60	$(3,3,3,3,5)$	92	$80:3+12:5$	150
16	Pseudorhombicuboctahedron	24	$(3,4,4,4)$	26	$8:3+18:4$	48

[9]. Before Kepler, partial lists were found by Piero della Francesca [24], Albrecht Dürer and W. Jamnitzer [9]. Pictures of some of the Archimedean solids made by Leonardo da Vinci (they served as illustrations for a book by Luca Pacioli) may be found in many books on history of mathematics. The last figure, the pseudorhombicuboctahedron, was overlooked for more than 2000 years and found only in the XX-th century [1].

With three exceptions (namely, polyhedra number 8, 9 and 16 in Table 1) the group of the orientation-preserving isometries acts transitively on the vertices of each polyhedron. In a more naive language we may say that its vertices are indistinguishable. The same does not hold for the truncated cuboctahedron (resp. truncated icosidodecahedron), whose vertex types, when read in positive direction, may be $(4,6,8)$ or $(4,8,6)$ (resp. $(4,6,10)$ or $(4,10,6)$). But the group of all isometries, orientation-preserving and reversing ones, acts transitively on vertices. (Apart from the vertex-transitive polyhedra there are 92 convex polyhedra whose faces are regular polygons, see [15], [32].)

Two *snub* figures have two 'chiral', or 'enantiomorphic' forms each. This means that they are not congruent to their mirror image. For our work this implies that their *fields of definition* are no longer $\mathbb{Q}$ but some imaginary quadratic fields. We will see later that in both cases the field is $\mathbb{Q}(\sqrt{-15})$.

For the pseudorhombicuboctahedron neither of the isometry groups acts transitively on its vertices. This might be the intuitive reason why it took so long to discover this polyhedron. For the same reason many authors do not include it in the list of Archimedean solids. It would be only too easy for us to follow their example, since for exactly the same reasons the computation of the corresponding Belyi function proved to be the most difficult. But we did not want to avoid the challenge. Curiously enough, the field of definition of this map is of degree 4; namely, it is $\mathbb{Q}(\sqrt[4]{12})$. It turns out that, besides the pseudorhombicuboctahedron, there exist three other (not semi-regular!) maps with the same vertex and face distributions and with the same group of orientation-preserving automorphisms, which thus form a Galois orbit containing 4 maps.

Remark 3.2. In the book [22] one may find the spherical coordinates of the vertices of all Archimedean polyhedra. The question is, if we project their edges from the center of the sphere onto its surface, and then make the stereographic projection on the plane, shall we obtain the corresponding dessins d'enfants? The answer is yes for Platonic solids and *no* for Archimedean ones, because in the dessins d'enfants all the angles at each vertex must be equal.

Now, returning to the idea so vividly expressed by Kepler (see epigraph), we may ask which of the two geometric forms is 'more authentic'. In this connection we would like to note that for three-dimensional polytopes there are infinitely many geometric realizations for any combinatorial type. Therefore, we must impose additional 'human made' conditions of regularity of all faces and of equality of all polyhedral angles. At the same time, the geometric form of the dessin d'enfant is unique (up to a linear fractional transformation), and this is true for any map, not only for regular or semi-regular ones.

The polyhedra dual to the Archimedean ones may also be considered as semi-regular. We do not treat them here for the obvious reasons given in Remark 2.6. The pictures of the polyhedra in the next section are borrowed from [2]. Many other pictures may be found in [31].

4. Computation of Belyi functions

Our methods may roughly be classified as follows:

- We used various symmetry and covering considerations and many relations that exist between different Platonic and Archimedean bodies. We also used some cartographic operations, such as 'vertex truncation' or 'edge duplication' (see below). All these operations may be represented as compositions of Belyi functions, see [10].
- When the map in question is reduced as much as possible, we use 'brute force' algorithms in order to compute its Belyi function. More precisely, this means (1) an efficient MAPLE package implemented by the first author for manipulating Belyi functions and producing dessins d'enfants; (2) the GB program for finding Gröbner bases [11,12,26]; and (3) an interface between the two systems, also implemented by the first author.

In what follows, $F_k(z)$ denotes the Belyi function for the map number k in Table 1, $k = 1, \ldots, 16$. Sometimes we omit expressions that are too complicated.

4.1. Prisms and antiprisms

The simplest type of a symmetry is the rotational one.

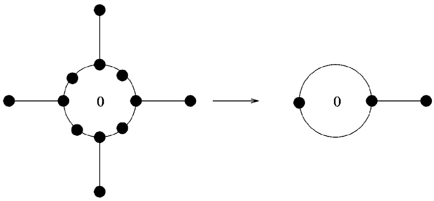

Fig. 6. Reducing a map using rotational symmetry.

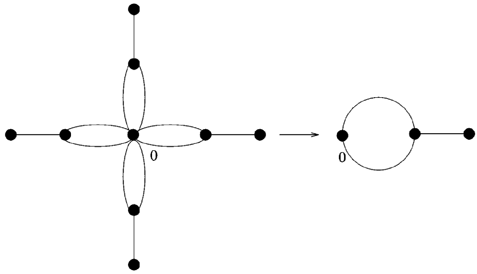

Fig. 7. Another example.

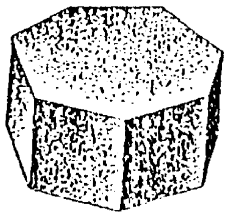

Fig. 8. The 7-prism.

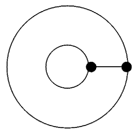

Fig. 9. Map (A): reduced prism.

Suppose we have a map possessing a center of symmetry of order n, and this center is positioned at the origin of the complex plane. Making the transformation $z \mapsto z^n$ we obtain a 'reduced map' which consists of only one 'branch' of the original map. It remains now to find the Belyi function for the simpler 'reduced' map.

Note that we may get the same map starting from different ones, as is shown in Figs. 6 and 7. This operation may be immediately used in order to find Belyi functions for prisms and antiprisms.

Prisms: The Belyi function of the map (A) of Fig. 9 is $f_{\mathrm{A}}(z) = -108(z+1)^4 z/(z^2 - 14z + 1)^3$, and $F_1^{(n)} = f_{\mathrm{A}}(z^n)$. Note that $F_1^{(4)}(z) = f_{\mathrm{cube}}(z)$ (see Figs. 8–10).

Antiprisms: The Belyi function of the map (B) is $f_{\mathrm{B}}(z) = -256(z+1)^3(8z-1)^3 z/(8z^2 - 20z - 1)^4$, and $F_2^{(n)} = f_{\mathrm{B}}(z^n)$ (Figs. 11–13).

4.2. Truncated platonic solids

For an arbitrary map, the operation of truncating the vertices consists of replacing every vertex of the map by a face having the same degree as the vertex. All the vertices of the new map thus obtained are of degree 3; see Fig. 14.

If we apply the Belyi function of the original map to the truncated map, we obtain as its image, instead of the segment $[1, \infty]$, a small 'truncating hypermap' (see Fig. 15), with a vertex of degree 3 inside the segment $[1, \infty]$, and with a face of degree 1 going 'around infinity'. It is easy to find its Belyi function

$$f_{\mathrm{trunc}}(z) = -27\frac{z^2}{(z-4)^3}.$$

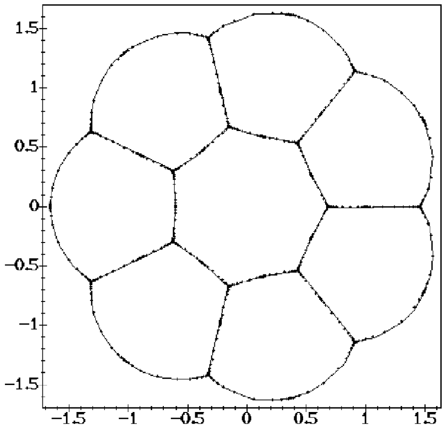

Fig. 10. Dessin for $F_1^{(7)}$.

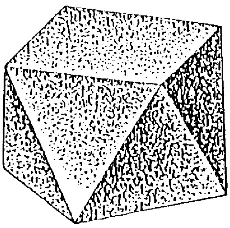

Fig. 11. The 4-antiprism.

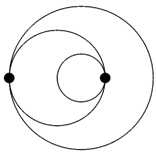

Fig. 12. Map (B): reduced antiprism.

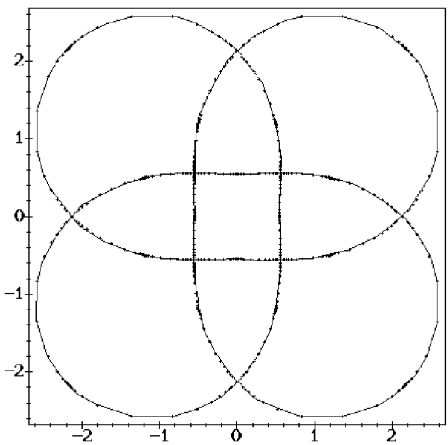

Fig. 13. Dessin for $F_2^{(4)}$.

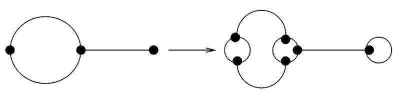

Fig. 14. The truncation operation.

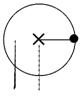

Fig. 15. The truncating hypermap.

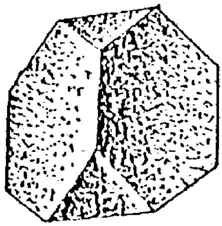

Fig. 16. The truncated tetrahedron.

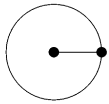

Fig. 17. Map (C): reduced tetrahedron.

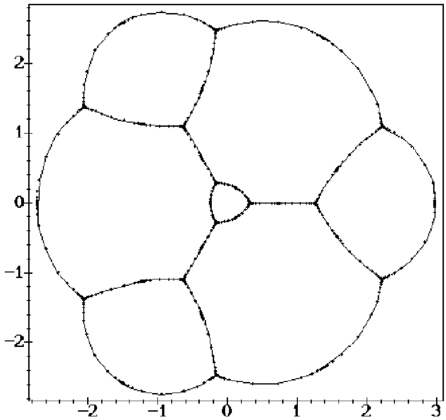

Fig. 18. Dessin for F_3.

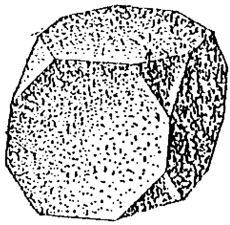

Fig. 19. The truncated cube.

Finally, if $f(z)$ is the Belyi function of a map, then $f_{\text{trunc}}(f(z))$ is the Belyi function of the corresponding truncated map.

Truncated tetrahedron (see Figs. 16–18): The map (C) of Fig. 17 is the reduced tetrahedron . We have

$$f_{\mathrm{C}}(z) = -64\frac{(z+1)^3}{z(z-8)^3}, \qquad f_{\text{tetra}}(z) = f_{\mathrm{C}}(z^3),$$

and thus

$$F_3(z) = f_{\text{trunc}}(f_{\text{tetra}}(z)) = 1728\frac{(z^3+1)^6 z^3 (z^3-8)^3}{(z^{12}-8z^9+240z^6-464z^3+16)^3}.$$

Truncated cube (see Figs. 19 and 20):

$$\begin{aligned} F_4(z) &= f_{\text{trunc}}(f_{\text{cube}}(z)) \\ &= \frac{19683}{4}\,\frac{(z^8-14z^4+1)^3(z^4+1)^8 z^8}{(z^{24}-15z^{20}+699z^{16}-2666z^{12}+699z^8-15z^4+1)^3}. \end{aligned}$$

Truncated octahedron (see Figs. 21–23): We have $f_{\mathrm{D}}(z) = 1/f_{\mathrm{A}}(z)$, hence $f_{\text{octa}}(z) = f_{\mathrm{D}}(z^4)$, and

$$\begin{aligned} F_5(z) &= f_{\text{trunc}}(f_{\text{octa}}(z)) \\ &= 2916\frac{z^4(z^4+1)^4(z^4-4z^2+1)^6(z^4+4z^2+1)^6}{(z^{24}+390z^{20}+2319z^{16}-236z^{12}+2319z^8+390z^4+1)^3}. \end{aligned}$$

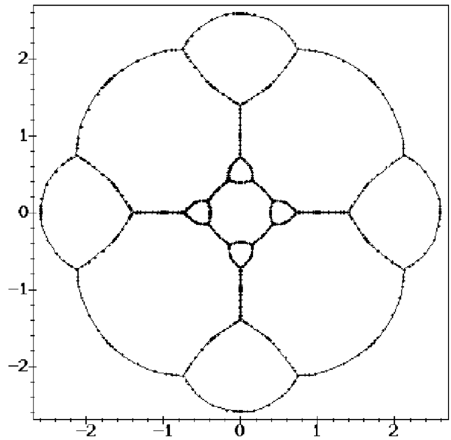

Fig. 20. Dessin for F_4.

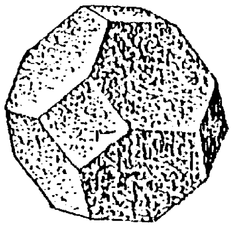

Fig. 21. The truncated octahedron

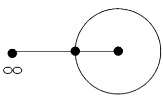

Fig. 22. Map (D): reduced octahedron (dual to the map (A)).

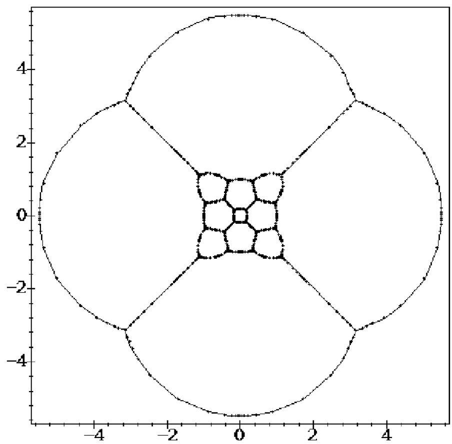

Fig. 23. Dessin for F_5.

Truncated icosahedron (see Figs. 24–26): For the map (E) of Fig. 25 we compute

$$f_{\mathrm{E}}(z) = \frac{1}{1728}\frac{(z^4+228z^3+494z^2-228z+1)^3}{(z^2-11z-1)^5 z}.$$

Therefore $f_{\mathrm{icosa}}(z) = f_{\mathrm{E}}(z^5)$, and $F_6(z) = f_{\mathrm{trunc}}(f_{\mathrm{icosa}}(z))$.

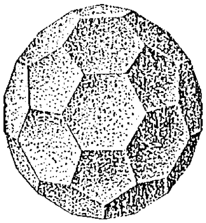

Fig. 24. The truncated icosahedron

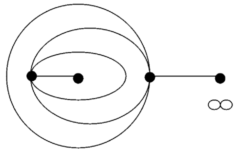

Fig. 25. Map (E): reduced icosahedron.

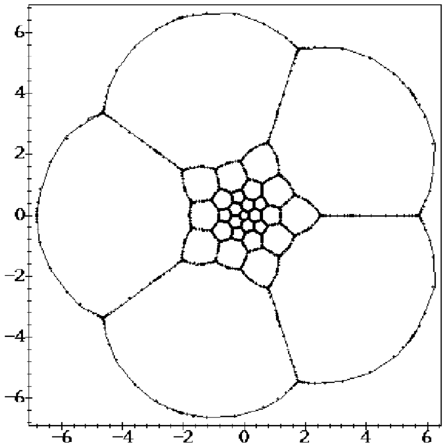

Fig. 26. Dessin for F_6.

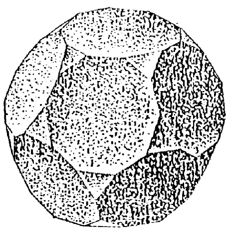

Fig. 27. The truncated dodecahedron.

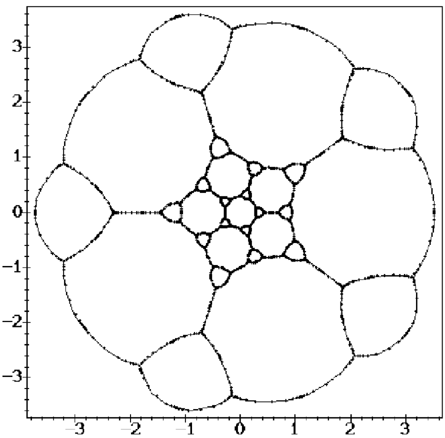

Fig. 28. Dessin for F_7.

Truncated dodecahedron (see Figs. 27 and 28): We have $F_7(z) = f_{\text{trunc}}(f_{\text{dodeca}}(z))$.

Remark 4.1. The symmetry of the truncated icosahedron is very interesting from the group theoretic point of view [20,21]. Together with the explicit formula for its Belyi function this may give rise to interesting examples for Galois theory. We think that other figures are equally interesting, but not yet thoroughly studied.

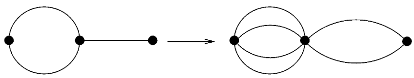

Fig. 29. The duplication operation.

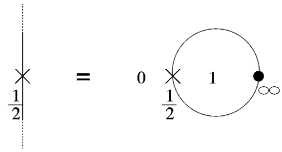

Fig. 30. The duplicating map.

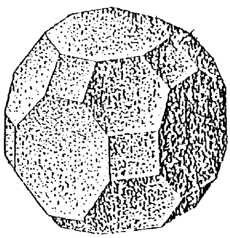

Fig. 31. The truncated cuboctahedron.

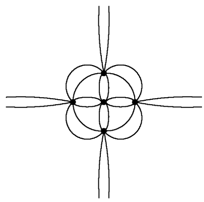

Fig. 32. Map (F): the duplicated octahedron.

4.3. Truncated cuboctahedron and icosidodecahedron

These two Belyi functions could have been computed in the same way as above (that is, using the truncation operation), if only we had known the Belyi functions for the cuboctahedron and icosidodecahedron themselves (before truncation). We choose another way out and introduce an operation of *duplication of edges*. This operation is also defined for arbitrary maps. It consists in replacing every edge by two edges forming a face of degree 2, as in Fig. 29. In the language of Belyi functions this operation consists in replacing a function f by the function $f_{\text{dupl}}(f)$, where f_{dupl} is the Belyi function for the map on Fig. 30, namely, $f_{\text{dupl}}(z) = 4z(1-z)$.

Truncated cuboctahedron (see Figs. 31–33): The map of the truncated cuboctahedron may also be considered as the truncation of the map (F) of the duplicated octahedron. Thus

$$F_8(z) = f_{\text{trunc}}(f_{\text{dupl}}(f_{\text{octa}}(z))).$$

See also Remark 4.2.

Truncated icosidodecahedron (see Figs. 34–36): In the same way, the truncated icosidodecahedron coincides with the truncation of the map (G) of the duplicated icosahedron. Thus

$$F_9(z) = f_{\text{trunc}}(f_{\text{dupl}}(f_{\text{icosa}}(z))).$$

4.4. Cuboctahedron and icosidodecahedron

Cuboctahedron (see Figs. 37 and 38): We have already computed the Belyi function F_8 for the *truncated* cuboctahedron. Now in order to get the Belyi function for the cuboctahedron itself, we must bring together the centers of all faces of degree 4 of F_8 (which become vertices of the new map), and in the same way bring together the centers of all the faces of degrees 8 and 6 (they become centers of faces of the new

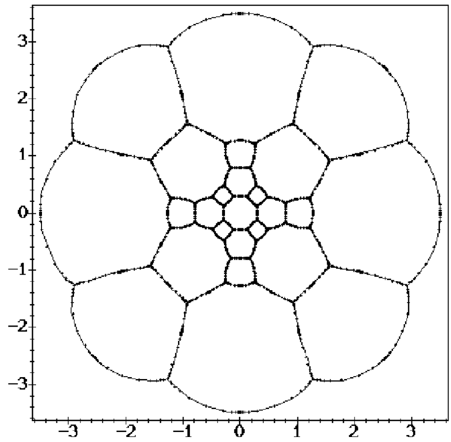

Fig. 33. Dessin for F_8.

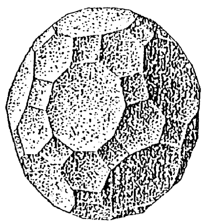

Fig. 34. The truncated icosidodecahedron.

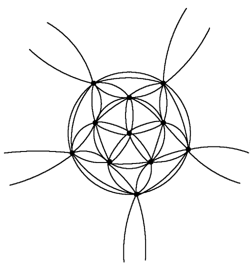

Fig. 35. Map (G): the duplicated icosahedron.

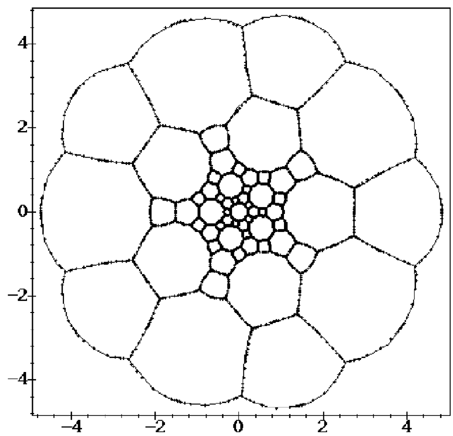

Fig. 36. Dessin for F_9.

map, of degrees 4 and 3 respectively). It remains to find the numeric factor in front of the function.

We can also obtain F_{10} if we try to 'draw the dessin on the cube' (Fig. 37). It is easy to guess that the vertices of the cuboctahedron occupy the same positions as middle

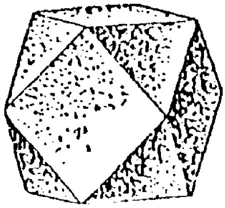

Fig. 37. The cuboctahedron.

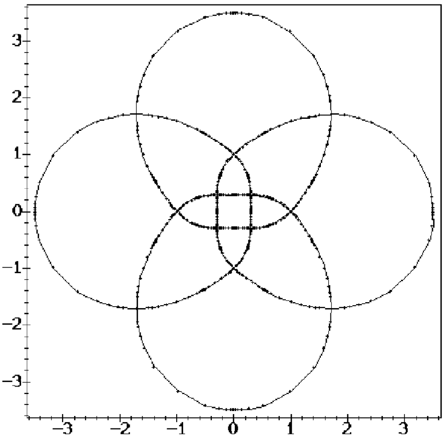

Fig. 38. Dessin for F_{10}.

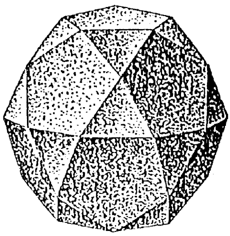

Fig. 39. The icosidodecahedron.

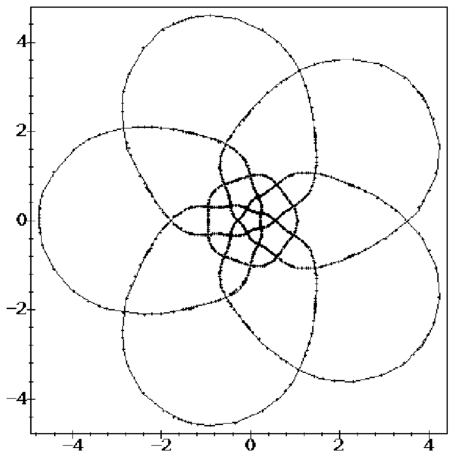

Fig. 40. Dessin for F_{11}.

points of the edges of the cube; the centers of faces of degree 4 of the cuboctahedron are at the centers of the cube faces; and the centers of the faces of degree 3 of the cuboctahedron are at the cube vertices. All this may be summed up as follows:

$$F_{10}(z) = -4\frac{f_{\mathrm{cube}}(z)}{(f_{\mathrm{cube}}(z)-1)^2} = 432\frac{(z^8-14z^4+1)^3(z^4+1)^4z^4}{(z^{12}+33z^8-33z^4-1)^4}.$$

Icosidodecahedron (see Figs. 39 and 40): Using the same techniques as above we get

$$F_{11}(z) = -6912\frac{(z^{20}+228z^{15}+494z^{10}-228z^5+1)^3(z^{11}-11z^6-z)^5}{(z^{30}-522z^{25}-10005z^{20}-10005z^{10}+522z^5+1)^4}.$$

Remark 4.2. The simple geometric fact that two different maps may produce the same map after truncation has a remarkable impact on the old problem of the uniqueness of

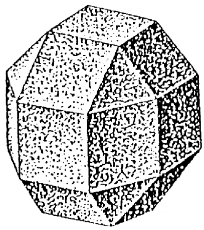

Fig. 41. The rhombicuboctahedron.

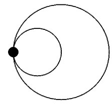

Fig. 42. Map (H).

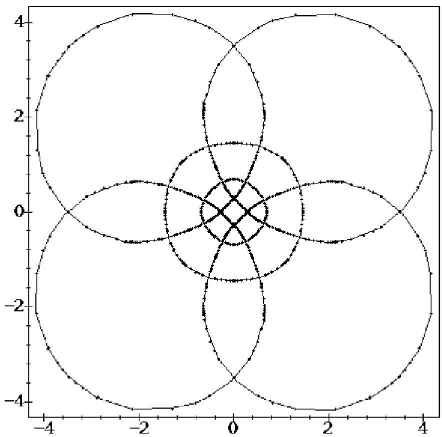

Fig. 43. Dessin for F_{12}.

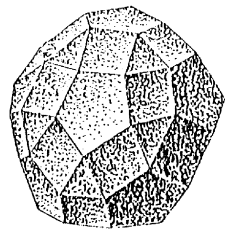

Fig. 44. The rhombicosidodecahedron.

representation of a rational function as a composition of simpler functions [16,25]. In the case of polynomials there are, roughly speaking, only two cases of non-uniqueness: when polynomials are of the form z^n or when they are Chebyshev polynomials. 'The analogous problem for fractional rational functions is much more difficult. There is a much greater variety of possibilities...' (Ritt [25]). Our example is very interesting because of the fact that $f_{\rm trunc}(F_{10}(z)) = f_{\rm trunc}(f_{\rm F}(z))$, while $F_{10}(z) \neq f_{\rm F}(z)$.

4.5. Two rhombi-polytopes

The rhombicuboctahedron (Fig. 41) has the same symmetry group as the cube. If we try to 'draw' it on the cube surface, we will see that all the positive triangles of the canonical triangulation of the cube contain the same image, and the same is true for the negative triangles. If we repeat both images only once, we get the map (H) of Fig. 42. This map is the image of the dessin of the rhombicuboctahedron under the function $f_{\rm cube}$ (Fig. 43). The same map is the image of the rhombicosidodecahedron under the function $f_{\rm icosa}$.

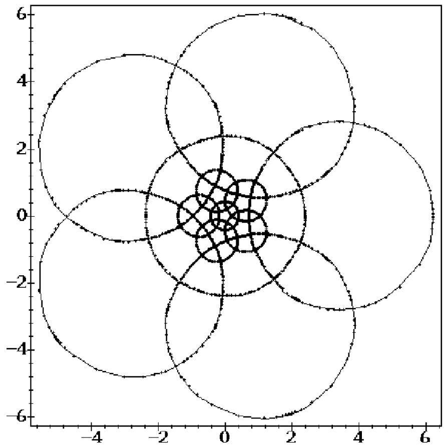

Fig. 45. Dessin for F_{13}.

The Belyi function for the map (H) is

$$f_{\mathrm{H}}(z) = 16\frac{(z-1)^2 z}{(z+1)^4}.$$

Rhombicuboctahedron:

$$F_{12}(z) = f_{\mathrm{H}}(f_{\mathrm{cube}}(z))$$
$$= -1728\frac{(z^8-14z^4+1)^3(z^{16}+34z^{12}-34z^4-1)^4 z^4}{(z^{24}-150z^{20}+159z^{16}-3476z^{12}+159z^8-150z^4+1)^4}.$$

Rhombicosidodecahedron (see Figs 44 and 45): $F_{13}(z) = f_{\mathrm{H}}(f_{\mathrm{icosa}}(z))$.

4.6. The snub polytopes

The group of the orientation-preserving automorphisms of the snub cube coincides with that of the cube itself. Hence we must try the same approach as in the previous subsection, that is, 'draw' the snub cube on the cube surface, and find its image via f_{cube}. The result now depends on our choice of 'left' or 'right' twin figure. In this way we obtain one of two hypermaps (I), (J), see Figs. 46 and 47. Their Belyi functions are algebraically conjugate:

$$f_{\mathrm{IJ}}(z) = \frac{25(3+8x)}{7496192}\frac{z(88z-(57x+64))^3}{(z+x)^5}$$

and

$$f_{\mathrm{IJ}}(z) - 1 = -\frac{1}{7744}\frac{(88z^2-(580x+256)z+(57x^2+64x))^2(z-1)}{(z+x)^5},$$

where x is a root of the equation $x^2 - \frac{7}{64}x + 1 = 0$, that is,

$$x = \frac{7 \pm 33\sqrt{-15}}{128}.$$

One of the roots gives the hypermap (I), the other one gives (J).

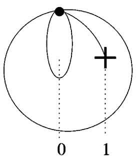

Fig. 46. Hypermap (I).

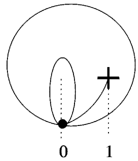

Fig. 47. Hypermap (J).

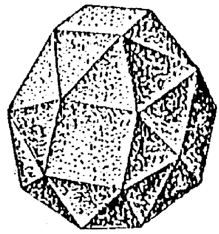

Fig. 48. The snub cube.

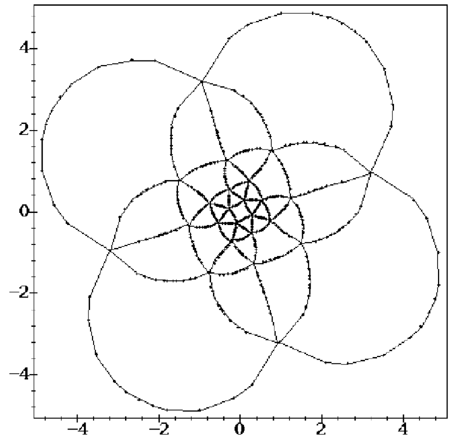

Fig. 49. Dessin for F_{14}.

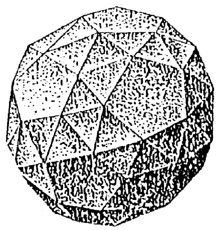

Fig. 50. The snub dodecahedron.

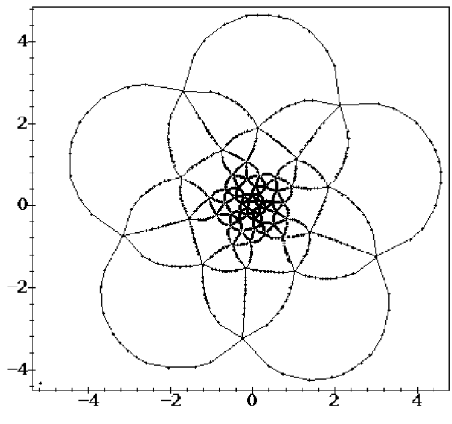

Fig. 51. Dessin for F_{15}.

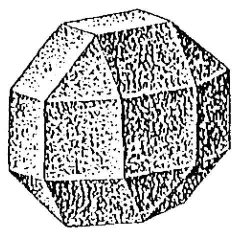

Fig. 52. The pseudorhombicuboctahedron.

The result is exactly the same if we apply f_{dodeca} to the snub dodecahedron.

Snub cube: $F_{14}(z) = f_{\mathrm{IJ}}(f_{\mathrm{cube}}(z))$ (see Figs. 48 and 49).

Snub dodecahedron: $F_{15}(z) = f_{\mathrm{IJ}}(f_{\mathrm{dodeca}}(z))$ (see Figs. 50 and 51).

We give the dessins for only one of two enantiomorphic forms of each figure.

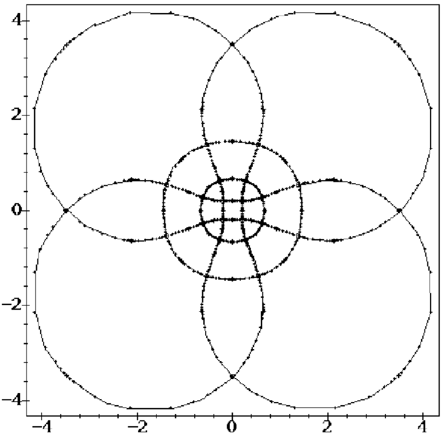

Fig. 53. Dessin for F_{16}.

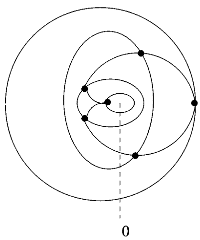

Fig. 54. Map (K): reduced pseudorhombicuboctahedron.

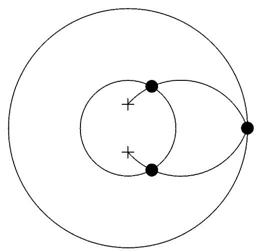

Fig. 55. Hypermap (L).

4.7. The pseudorhombicuboctahedron

The computation is very complicated, so we only outline the main stages. First, the transformation $z \mapsto z^4$ applied to the dessin of Figs. 52 and 53 gives the map (K) of Fig. 54. Using a kind of an antipodal symmetry of the map (K), we may reduce it once more, thus obtaining the hypermap (L) (Fig. 55). It remains to compute the Belyi function f_L for the hypermap (L).

Surprise: This hypermap has three other Galois conjugates, and its Belyi function is defined over the field of degree 4, namely, $\mathbb{Q}(\sqrt[4]{12})$; we mean by that the field generated by all the four roots of degree 4 of 12.

(For the fields of degree greater than 2 it is not an easy thing to compare two different presentations of the same field. Our own computations gave us more complicated expressions. The above much simpler form of the field was found by Granboulan [13].)

What does it all mean?

The answer is, there exist, besides the pseudorhombicuboctahedron, three other maps with the same vertex distribution (namely, 4^{24}), with the same face distribution (namely, $4^{18}3^8$), and with the same group of orientation-preserving automorphisms (namely, the cyclic group of order 4). These maps are shown in Fig. 56 (for two of them,

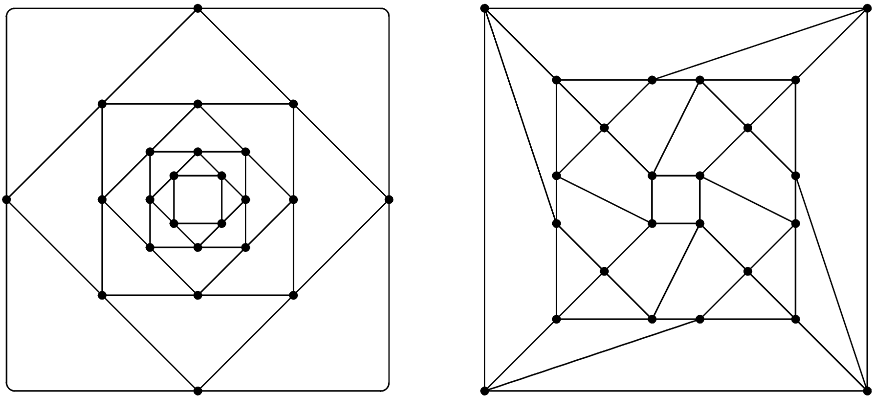

Fig. 56. Maps conjugate to the pseudorhombicuboctahedron.

corresponding to the imaginary roots of $\sqrt[4]{12}$, only one of the two enantiomorphic forms is given). The four Belyi functions have the same expression, but their coefficients depend on $\sqrt[4]{12}$. Substituting four different values of the root gives four different Belyi functions, thus producing the four maps in question.

The right-hand map of Fig. 56 may be easily found in the following way: take a truncated octahedron; cut its 8 hexagonal faces by a diagonal into two quadrilaterals, thus obtaining 16 quadrilaterals; cut 4 of its 6 quadrilateral faces by a diagonal into two triangles, thus obtaining 8 triangles; the remaining 2 quadrilateral faces are not touched. There are exactly two ways of doing that so that all the vertex degrees become equal to 4.

According to the well-known theorem of Steinitz [30], any 3-connected planar map without loops and multiple edges and without vertices of degree 1 and 2, can be realized as a polyhedron. However, the faces of the above three maps cannot be regular polygons, because there are some vertices surrounded by four quadrilaterals.

A natural question arises: *Do there exist*, *for other Archimedean solids*, *non semi-regular maps having the same vertex and face distributions*? Probably the answer is well known; but we did not find the appropriate references.

Acknowledgements

The authors are grateful to Alberto Marini for interesting historical remarks, and to Jonathan Borwein for the information concerning the relations of Belyi functions to theta functions (see Remark 2.7). The discussion of the computation of Belyi function for the hypermap of Fig. 55 with Louis Granboulan was very helpful. Finally, we are grateful to the unknown referee for very useful suggestions concerning the text of the paper.

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